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AdS gravitational waves in string theory

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Abstract : *AdS* gravitational wave solutions and some of their properties, in pure gravity theory, are briefly reviewed Similar solutions and their symmetry properties are then discussed in string and M-theory, as classical solutions of low energy effective actions

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1 Brief review of results in pure gravity

Study of gravitational waves, both theoretically as well as experimentally, has been a topic of research for long time due to its far reaching implications In this article, following [1] (see also [2–5]), we first review gravtiational waves and their properties in *AdS* backgrounds String and M-thoery generalizations are presented in the later sections

Theoretically, a specific exact solution of Einstein equation of such a type was found in 1963 by Kaigodorov A general metric, however, was presented in 1985 [5] This gravitational wave metric has a form

$$ds^2 = \frac{q^2}{u^2}\left(du^2 + dx^2 + 2dx^+ dx^- + H(u, x^+, x)dx^{+2}\right), \quad (1.1)$$

where $x^{\pm}$ are two null directions, x is a spatial direction and u represents the *AdS* radial coordinate Also, $H = 0$ corresponds to AdS_4 metric in 'Poincare coordinate' In above eq q^2 defines the *AdS* radius and is determined in terms of the cosmological constant Λ in Einstein equation by a relation $q^2 = -\Lambda$, with $\Lambda < 0$ for an *AdS* space $u = 0$ is the AdS boundary and $u = \infty$ corresponds to the AdS horizon The general form of the *AdS* gravitational wave, in eq (1.1), when inserted into Einstein field equation results into

$$H_{uu} - \frac{2}{u}H_u + H_{xx} = 0 \quad (1.2)$$

This wave equation determines the wave profile H which is an arbitrary function of x^+

Due to the time dependence of the wave profile $H(x^+)$, waves lasting finite amount of time are possible In particular, $H(x^+) = \delta(x^+)$ represents an impulsive wave Explicit solutions of *AdS* gravitational waves have also been written, by solving the wave-eq (1.2) One obtains

$$H = u^2 \frac{\partial}{\partial u}\left(\frac{f + \bar{f}}{u}\right) = \frac{1}{2}(f_{,\zeta} + \bar{f}_{,\bar{\zeta}})(\zeta + \bar{\zeta}) - (f + \bar{f}), \quad (1.3)$$

where $f(\zeta), \bar{f}(\bar{\zeta})$, are arbitrary functions of $\zeta(\bar{\zeta})$, with $\zeta = u + ix$. Now, to find out the nature of the resulting space-time, as well as the gravitational wave propagation in them, one analyzes the relevant 'geodesic equations' as well as the 'equations of geodesic deviation' The first one gives the list of allowed trajectories while the later sheds light on the properties of the gravitational waves

We start with the discussion of the geodesic deviation equations (GDE) These equations are derived by variations of geodesic equation and describe the motion of a displacement vector connecting two neighboring "time-like" geodesics. In other words, the motion of a displacement vector connecting test particles on nearby trajectories can be found out by studying GDE The nature of the gravitational field is then analyzed through their influence on particle motion In mathematical terms, for z^μ a displacement vector, one has

$$\frac{D^2 Z^\mu}{d\tau^2} = -R^\mu_{\alpha\beta\gamma} \frac{dx^\alpha}{d\tau} Z^\beta \frac{dx^\gamma}{d\tau}, \tag{1 4}$$

where τ is the affine parameter, $x^\mu(\tau)$ parameterizes a particular trajectory and $dx^\alpha/d\tau$ denotes the 4-velocity Also, in the above equation, covariant derivative 'D' along a trajectory is defined as

$$\frac{DA^\mu}{d\tau} = \frac{dA^\mu}{d\tau} - \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} A^\rho \tag{1 5}$$

In analyzing the geodesic deviation equations, one's aim is to attempt at making coordinate independent statements For example, whether gravitational wave is present or not in a given situation, should not depend on the choice of coordinates In view of this, it is more suitable to turn the vectorial geodesic deviation eq (1 4) into a set of scalar ones by projecting on to a frame which is parallely transported along the geodesic For this purpose, one introduces a set of orthonormal vectors, $e^{(a)\mu}$, $(a = 0, 1, 2, 3)$ $e^{(a)}.\ e^{(b)} = \eta^{ab}$ and $e^{(0)\mu} = t^\mu$ is the tangent vector to the curve $x^\mu(\tau)$ · $dx^\mu/d\tau = t^\mu$. In addition, frame vectors $e^{(i)\mu}$, $(i = 1, 2, 3)$ are covariantly constant

$$\frac{De^{(i)\mu}}{d\tau} = 0 \tag{1 6}$$

Contraction of indices of the vectorial geodesic deviation eq (1 4), with the frame vectors $e^{(i)\mu}$, t^μ, defines the projection on the frame Now as an example one first observes that the choice

$$H(u, x, x^+) = u^3, \tag{1 7}$$

satisfies the wave eq (1 2) The corresponding metric describes the Kaigodorov space-time

$$ds^2 = \frac{q^2}{u^2}\left(du^2 + dx^2 + 2dx^+dx^- + u^3 dx^{+2}\right) \tag{1 8}$$

Kaigodorov metric has singularities at $u = 0$ and $u = \infty$ It can be verified that curvature invariants of the type R, $R_{\mu\nu}R^{\mu\nu}$ etc are all regular at $u = 0$ and it is therefore only a coordinate singularity At $u = \infty$, on the other hand, certain invariants constructed from $R_{\mu\nu\rho\sigma}$ and frame vectors $e^{(a)\mu}$'s, in particular their combination appearing in the RHS of the GDE (1 4), diverge This is known as a PP-curvature singularity It is interpreted as the presence of an infinite tidal force at $u = \infty$, where the AdS horizon is located The presence of PP-curvature singularities also hold for a generic wave profile H

As stated earlier, quantities such as

$$R_{(0)(i)(0)(j)} \equiv R_{\mu\nu\rho\sigma} t^\mu e^{(i)} t^\rho e^{(j)\sigma}, \tag{1 9}$$

appearing in the RHS of geodesic deviation equation blow up at the PP-curvature singularity An analysis of the structure of $R_{(0)(i)(0)(j)}$ also determines the nature of the particular gravitational wave one is studying For example, for a wave profile of the type

$$H(u, x, x^+) = f(x^+) + g(x^+)x + h(x^+)(u^2 + x^2), \tag{1 10}$$

which corresponds to $f_{\zeta\zeta\zeta} = 0$ in eq (1 3), it can be shown that $R_{(0)(i)(0)(j)}$'s reduce to a simple equation for the displacement vector

$$Z^{(i)} = \frac{\Lambda}{3}, (i = 1, 2, 3), \Lambda < 0 \tag{1 11}$$

and corresponds to the equation of the displacement vector in a pure AdS space In other words, H in eqn (1 10) does not qualify as a genuine gravitational wave

Further analysis of the geodesic deviation equation for generic H, including that of Kaigodorov type, implies that the motion of the particles is governed by 2 independent amplitudes, affecting their transverse motion In other words, the number of physical polarizations associated with the above classical solutions is that of the gravity waves In addition, the amplitudes of particle motion has a periodicity π, thus implying that they correspond to helicity-2 objects

We have therefore described the geodesic deviation equations in the Siklos metric back-ground (1 1) and confirmed that they correspond to the gravitational wave propagation in AdS backgrounds Geodesic equations, on the other hand, are harder to analyze for arbitrary H However, they can be solved for the Kaigodorov metric $H = u^3$. We have earlier pointed out that $u = 0$ corresponds to a coordinate singularity while $u = \infty$ is a PP-curvature singularity Geodesic observers moving along certain time like geodesic reach the singularity at $u = \infty$ in a finite proper time On the other hand, $u = 0$ is not reached by any time-like observer Space-time thus splits into disjoint sets, $u > 0$ and $u < 0$, having different properties in the two regions To conclude, in this section, we have reviewed that Siklos solutions describe exact gravitational waves propagating in AdS universe.

2. *AdS* gravitational waves in string and M-theory

String and M-theory AdS gravitational waves have been discussed previously [6–8]. In particular, in [7] attempt has been made to study them in the context of wave propagation on non-dilatonic branes, such as $D3$, $M2$, $M5$

A 'near horizon limit' on these backgrounds then gives the AdS gravitational wave solution in string as well as M-theory depending on the choice of the non-dilatonic brane that is used. This brane wave solution has a metric of the form

$$ds^2 = f^{-2/d}(r)\left(dx^+dx^- + \sum_i dx^{i^2} + H(r,x^+,x^i)dx^{+^2}\right) + f^{2/\tilde{d}}(r)\left(dr^2 + r^2 d\Omega_{\tilde{d}+1}^{\ 2}\right) \quad (2.1)$$

with $f = 1+\frac{k}{r^d}$ $d = 4$, $\tilde{d} = 4$ corresponds to the brane-wave on a $D3$ brane while, $d = 6(3)$, $\tilde{d} = 3(6)$ are the brane-wave solutions on $M5$ ($M2$) Also, $H = 0$ reduces to the static brane solutions The AdS gravitational wave metric of the last section is obtained by taking $r \rightarrow 0$ limit of the $d = 3$, $\tilde{d} = 6$ case in (2.1) and then identifying $k = q$ with a coordinate change $r \rightarrow u$, given explicitly below

For the string and M-theory solutions, the above metric is also accompanied by a p-form flux They are listed for different branes below

$$D3 \quad F^{(5)} \sim dx^+ \wedge dx^- \wedge dx^1 \wedge dx^2 \wedge df^{-1} + (*), \quad (2.2)$$

$$M2 \quad F^{(4)} \sim dx^+ \wedge dx^- \wedge dx \wedge dx^2 \wedge df^{-1}, \quad (2.3)$$

$$M5 \quad *F^{(4)} \sim dx^+ \wedge dx^- \wedge dx^1 .. \wedge dx^4 \wedge df^{-1}, \quad (2.4)$$

with $(*)$ in eq (2.2) representing the Hodge dual of the first term.

Metric in (2.1) and fluxes given above satisfy the relevant equations of motion for string or M-theory, provided once again H satisfies a wave-equation It is important to note that p-form fluxes given above are independent of H and therefore same as for the static branes case. In the AdS limit, $r \rightarrow 0$ the role of the flux is to compensate for the cosmological constant Λ in pure gravity and the brane charge determines the AdS radius In the near horizon limit, equation for H is identical to that of the pure AdS gravity in different dimensions, namely AdS_4, AdS_5 and AdS_7, for $M2$, $D3$ and $M5$ branes respectively, and matches with the notations in the previous section after making a coordinate transformation .

$$u = \frac{d}{\tilde{d}} k^{1/2} r^{(1-\tilde{d})/2} \quad (2.5)$$

Also, for $\tilde{d} > 1$, $r \rightarrow 0$ (or $u \rightarrow \infty$) represents the AdS horizon, whereas $r \rightarrow \infty$ or ($u \rightarrow 0$) represents the AdS boundary

Properties of these AdS waves have also been examined in some detail and the conclusions are similar to that for the pure gravity. There are no non-singular PP-wave in AdS^{d+1}, except for those profiles H which vanish fast enough, as one approaches the AdS horizon Note that Kaigodorov solution $H \sim u^d$, is singular at the horizon, as expected Killing symmetries of the gravitational waves are also known For $H = u^d$, the Killing vectors are

$$K^+ = \frac{\partial}{\partial x^+}, \quad k = \frac{\partial}{\partial x^+}, \quad K^i = \frac{\partial}{\partial x^i}, \quad L_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}, \quad (2.6)$$

and

$$J = u\frac{\partial}{\partial u} - \frac{(d+2)}{2}x^-\frac{\partial}{\partial x^-} + \frac{(d-2)}{2}x^+\frac{\partial}{\partial x^+} - x^i\frac{\partial}{\partial x^i} \quad (2.7)$$

$K^{\pm}$, K^i, L_{ij} is not only the symmetry of the AdS pp-wave, but also that of the brane-wave which gives the corresponding AdS pp-wave in the the near horizon limit Isometry J, on the other hand, is a symmetry only in this limit String and M-theory pp-wave solutions also preserve some amount of supersymmetry For brane-wave, these are the intersections of the brane supersymmetry conditions as well as the pp-wave supersymmetry condition, $\Gamma^+\epsilon = 0$ Finally, although K^+ is a Killing vector, but unlike in flat space, K^+ is not covariantly constant As a result, the usual argument about the pp-wave in flat space being an all-order solution of the string equations of motion, does not work for AdS pp-wave We now generalize the results of this section further to those with new wave profiles

3. String generalization with fluxes

As pointed out earlier, string and M-theory AdS solutions presented in the previous section were identical to the ones in pure gravity They had identical metric as well as the equations satisfied by the wave-profiles were also identical In this section, we present a more general class of AdS gravitational wave solutions, which allows for new possibilities of gravitational wave profiles.

In our new ansatz [9,10], the metric remains same as in previous solutions However, the p-form backgrounds are now different Additional null p-form fluxes are now turned on leading to the possibility of obtaining new wave profiles Explicit solutions for the wave profiles are also given, by making use of the known D-brane solutions in pp-wave backgrounds. One can find explicit examples of AdS pp-wave solutions in backgrounds : $AdS_3 \times S^3$, $AdS_5 \times S^5$, $AdS_7 \times S^4$, $AdS_4 \times S^7$ In this article, however, we present first the $AdS_3 \times S^3 \times R^4$ solutions. Later on we give a non-trivial M-theory generalization of the AdS_4 gravitational waves. Various other examples of gravitaional wave solutions in AdS_5 and AdS_7 backgrounds can be found in [10]. For interested readers, a more complete list of references can also be found in [10] and [7].

3.1. $AdS^3 \times S^3$ pp-wave :

We now begin [9] by writing down the ansatz for the gravitational wave solution in ten dimensional type IIB string theory on $AdS_3 \times S^3 \times R^4$. The metric is of a form

$$ds^2 = q\left\{\frac{du^2}{u^2}+\frac{1}{u^2}\left(2dx^+dx^- + H(u,x^+)dx^{+2}\right) + d\Omega_3^2\right\}+\sum_{i=1}^{4} dx^{i^2} \tag{3.1}$$

For the discussion below, we also parametrize the metric on S^3 as

$$d\Omega_3^2 = d\theta^2 + \cos^2\theta d\phi^2 + \sin^2\theta d\psi^2 \tag{3.2}$$

Additional field strengths turned on are the RR 3-form field strength

$$F^{(3)} = \pm\frac{2q}{u^3}dx^+\wedge dx^-\wedge du \pm 2q\sin\theta\cos\theta d\theta\wedge d\phi\wedge d\psi, \tag{3.3}$$

and NS-NS 3-form field strength

$$H^{(3)} = dx^+\wedge\left(\left(\frac{q^2A_a(x^a,x^+)}{u^3}\right)du\wedge dx^a + \left(\frac{q^2B_{ab}(x^a,x^+)}{u^2}\right)dx^a\wedge dx^b\right), \tag{3.4}$$

with indices a, b = 1, 2, 3 denoting the three S^3 coordinate θ, ϕ and ψ. The role of $F^{(3)}$ above is to compensate for $AdS_3 \times S^3$ curvature in the gravitational field equation, whereas a nontrivial 3-form $H^{(3)}$ modifies the wave profile H and can also be representing a wave propagation of the associated 2-form field.

The conditions following from the string effective action are

$$-\frac{H_{uu}}{2}+\frac{1}{2}\frac{H_{,u}}{u} = \frac{q^2}{u^4}\left(\frac{1}{2}A_aA^a + \frac{1}{4}B_{ab}B^{ab}\right), \tag{3.5}$$

$$\nabla^a A_a = 0, A_a + \frac{1}{2}\Delta^b B_{ab} = 0, \tag{3.6}$$

and

$$\partial_a A_b - \partial_b A_a = -2B_{ab} \tag{3.7}$$

To compare with the gravitational wave solutions in flat space, analogous condition for the pp-wave solution in flat space, mentioned earlier, has a form [13] .

$$\partial^I\partial_I H(x^+,x^I) + A_{IJ}A^{IJ} + 4\partial_{x^+}^2\phi(x^+) = 0, \tag{3.8}$$

with ϕ being the dilaton. The similarity of the above condition (3.5) can also be seen with anti-de Sitter gravitational waves in four dimensional gravity [1], by setting A_a's and B_{ab}'s to zero.

Now, a solution can be obtained by considering $A_a = B_{ab} = 0$, so that NS-NS three-form field $H^{(3)}$ is trivial. Then one only has a non-trivial R-R three-form flux given by eq. (3.3), in addition to the metric (3.1). Moreover, the R-R three-form flux is identical to the one in the case of $AdS^3 \times S^3$ background. This choice of A_a and B_{ab} already satisfies eqs. (3.6) and (3.7). Eq. (3.5) in this case implies a general wave profile given by

$$H(u,x^+) = f(x^+)u^2 + g(x^+), \tag{3.9}$$

with f and g being functions of x^+ only. Later on we will also discuss the supersymmetry property of this solution.

We now write down another gravitational wave solution by taking a scaling limit on a known $D1-D5$ brane in a pp-wave background [11]. In this case, the gravitational wave profile turns out to be independent of x^+ (similar to the case of Hpp-waves appearing in the case of 'BMN' duality in string theory), although further generalization of this solution to include x^+ dependence is possible and will also be discussed below.

In fact, it can be verified that the choice of A components

$$A_\theta = 0,\ A_\phi = 2\mu\cos^2\theta,\ A_\psi = 2\mu\sin^2\theta, \tag{3.10}$$

and nonzero B_{ab} components .

$$B_{\theta\phi} = \mu\sin 2\theta,\ B_{\theta\psi} = -\mu\sin^2\theta, \tag{3.11}$$

satisfy eqs. (3.6) and (3.7). The parameter μ in the above equation characterizes the gravitational wave. When $\mu = 0$ one reduces to $AdS_3 \times S^3$. Finally, eq. (3.5) is also satisfied for

$$H = -\frac{\mu^2q^2}{\ } \tag{3.12}$$

Eqs. (3.10), (3.11) and (3.12), together with the metric and 3-form fields in eqs. (3.1), (3.3) and (3.4), give a gravitational wave solution in type IIB string theory on $AdS_3 \times S^3 \times R^4$. We now show the connection of this solution to the $D1-D5$ branes in a pp-wave background in type IIB string theory.

The $D1-D5$ brane solution, reducing to the above gravitational wave on $AdS_3 \times S^3 \times R^4$ in a scaling limit, is given as [11]

$$ds^2 = (f_1 f_5)^{-1/2}\left(2dx^+ dx^- - \mu^2 \sum_{i=1}^{4} x_i^2 (dx^+)^2\right)$$

$$+\left(\frac{f_1}{f_5}\right)^{1/2} \sum_{a=5}^{8} (dx^a)^2 + (f_1 f_5)^{1/2} \sum_{i=1}^{4} (dx_i)^2,$$

$$e^{2\phi} = \frac{f_1}{f_5}, \quad H^{(3)}_{+12} = H^{(3)}_{+34} = 2\mu,$$

$$F^{(3)}_{+-i} = \partial_i f_1^{-1}, \; F^{(3)}_{mnp} = \epsilon_{mnpl}\, \partial_l f_5, \tag{3.13}$$

where f_1 and f_5 are the Green functions, in common transverse directions $x^1, \ldots, x^4$, representing $D1$ and $D5$ branes respectively $f_1 = 1 + \frac{q_1}{r^2}$, $f_5 = 1 + \frac{q_5}{r^2}$ and r is the radial coordinate in the transverse direction. In the scaling limit one takes (see for example [14]) $r \to 0$, in addition to setting $q_1 = q_5 = q$. It can now be verified that such a scaling limit, together with a coordinate change $r = q/u$, gives the gravitational wave solution of eqs (3.1), (3.3) and (3.4) when the wave profile is chosen as in eqs (3.10), (3.11) and (3.12). We have therefore shown that $D1-D5$ brane solution in pp-wave background gives, in a scaling limit, a gravitational wave in string theory in $AdS_3 \times S^3 \times R^4$ background.

Several other gravitational wave solutions in $AdS_3 \times S^3 \times R^4$ background can be constructed using 'duality' symmetries of string theory. The simplest ones of these comes from a direct application of S-duality on our general solution in eqs (3.1), (3.3) and (3.4). This symmetry leaves the metric unchanged, however, NS-NS and R-R 3-form fields are interchanged among themselves under the symmetry transformation. The specific wave profiles of both the solutions mentioned in the paper, namely the ones in (3.9) and (3.10)–(3.12), remain unchanged under the symmetry transformation. We do not write these solutions explicitly.

A more non-trivial example involves several applications of S and T-dualities, amounting to taking a scaling limit on another $D1-D5$ brane solution [11], involving now self-dual R-R 5-form field strengths ($F^{(5)}$) of the IIB string theory, rather then NS-NS 3-forms $H^{(3)}$ as in equation (3.13). The gravitational wave solution obtained by taking a scaling limit has the identical metricand 3-form flux as in eqs (3.1) and (3.3). However, one now has a (self-dual) R-R 5-form flux (in place of NS-NS 3-form of eq (3.4))

$$F^{(5)} = dx^+ \wedge \left(\left(\left(\frac{q^2 A_a(x_a, x^+)}{u}\right) du \wedge dx^a\right.\right.$$

$$\left. + \left(\frac{q^2 A_a(x_a, x^+)}{u}\right) dx^a \wedge dx^b\right)$$

$$\wedge \left(dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right), \tag{3.14}$$

and, as already mentioned earlier, $x^1, \ldots, x^4$ are the coordinates of R^4. The wave profile for this gravitational wave solution is identical to the one in (3.10), (3.11) and (3.12).

Further generalization of the gravitational wave solution, as obtained above from the $D1-D5$ branes in eqs (3.10)–(3.12), can be incorporated in string theory. It is evident from eqs (3.5), (3.6) and (3.7) that one can obtain a general class of explicit solutions by multiplying H in eq (3.12) by any function $F(x^+)$, provided A_a and B_{ab} in eqs (3.10) and (3.11) are multiplied by another function $G(x^+)$, satisfying $F(x^+) = G(x^+)$. Supersymmetry analysis for these x^+ dependent solutions proceeds in a similar fashion as presented above. The amount of supersymmetry also turns out to be identical.

As discussed in an earlier section, there are other papers [7,8] where gravitational waves in anti-de Sitter spaces have been discussed. However, as already mentioned, these gravitational waves correspond, in our language, to the situation when $A_a = B_{ab} = 0$ in our eqs (3.4) and (3.14). In particular, our $A_a = B_{ab} = 0$ solution (3.9) has been given earlier in [7] and shown to be equivalent to the pure AdS_3 without a gravitational wave. However, it is emphasized that the general situation above represents new gravitational wave structure in anti-de Sitter backgrounds with a non-trivial NS-NS 3-form (or R-R 5-form) flux mixing AdS_3 and S^3 spaces. We now present an example of the gravitational wave in a four dimensional AdS space.

3.2 AdS4 x S^7 Solution

Gravitational waves in AdS_4 backgrounds are of particular interest, due to their connection with the physics in four dimensions. In pure gravity theory, the gravitational waves in such backgrounds require the presence of cosmological constant term. In eleven dimensional M-theory that we are considering, one does not have any such cosmological constant term and the background AdS_4 is accompanied by a flux on S^7 in order to compensate for the opposite

Ricci curvature terms Phenomenological consequences of such a gravitational wave in $AdS_4 \times S^7$ background will be also of interest to examine along the lines of [4]

We now give an example of a gravitational wave in $AdS_4 \times S^7$ background Later on, in this section, we also show the connection of our solution with certain supersymmetric 'localized' M2 branes of [12] in the same way as was done above for other branes The metricis now written as

$$ds^2 = \frac{q}{4}\left\{\frac{du^2}{u^2} + \frac{1}{u^2}\left(2dx^+ dx^- + H(u, x^+, x, x^a)\right.\right.$$

$$\left.\left.\times dx^{+2} + dx^2\right) + 4d\Omega_7^2\right\}, \tag{3.15}$$

The 4-form flux is of the form

$$F^{(4)} = \frac{3q^{3/2}}{4} dx^+ \wedge dx^- dx \wedge du - \frac{\mu q^{9/4}}{4\sqrt{2}} \frac{A_{ab}}{u^{5/2}} dx^+$$

$$\wedge du \wedge dx^a \wedge dx^b + \frac{\mu q^{9/4}}{2\sqrt{2}} \frac{A_{abc}}{u^{3/2}} dx^+ \wedge dx^a \wedge dx^b \wedge dx^c, \tag{3.16}$$

where powers of q are chosen appropriately to have q-independent solution for A_{ab}'s etc below

Equations of motion then simplify to

$$\frac{1}{\sqrt{g}}\partial_c\left(\sqrt{g}A^{abc}\right) + 5A^{ab} = 0, \quad \frac{1}{\sqrt{g}}\partial_b\left(\sqrt{g}A^{ab}\right) = 0, \tag{3.17}$$

and

$$-\frac{H_{uu}}{2} + \frac{H_u}{u} - \frac{1}{2}H_{,i}^{,i} - \frac{1}{8u^2\sqrt{g}}\partial_a\left(\sqrt{g}\partial^a H\right)$$

$$= \frac{1}{32}\frac{\mu^2 q^{3/2}}{u^3}\left(\sum_{a,b} A_{ab}A^{ab} + \frac{1}{3}\sum_{a,b,c}(A_{abc}A^{abc}\right) \tag{3.18}$$

Bianchi identity gives

$$A_{abc} = \partial_{[a}A_{bc]} \tag{3.19}$$

One can also write an explicit solution for all the conditions, namely eqs (3.17), (3.18), and (3.19) For this we write down a metricon S^7 as

$$ds^2 = d\theta^2 + \cos^2\theta\left[d\phi^2 + \cos^2\phi d\psi^2 + \sin^2\phi d\omega^2\right]$$

$$\sin^2\theta\left[d\gamma^2 + \sin^2\gamma d\eta^2 + \sin^2\gamma d\beta^2\right]. \tag{3.20}$$

A simple embedding of the solutions of pure AdS gravity in four dimensions, to M-theory, can then be obtained by realizing those as $A_{ab} = A_{abc} = 0$ solutions of eqs (3.17) – (3.19)

Our new solutions for A_{ab} and A_ab_c are

$$\begin{aligned}
A_{\psi\theta} &= \cos\theta \cos^2\phi \cos\gamma, \\
A_{\omega\theta} &= \cos\theta \sin^2\phi \cos\gamma, \\
A_{\psi\gamma} &= -\cos^2\theta \cos^2\phi \sin\theta \sin\gamma, \\
A_{\omega\gamma} &= -\cos^2\theta \cos^2\phi \sin\theta \sin\gamma, \\
A_{\phi\psi} &= -\cos^2\theta \sin\theta \sin\phi \cos\phi \cos\gamma, \\
A_{\phi\omega} &= -\cos^2\theta \sin\theta \sin\phi \cos\phi \cos\gamma,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
A_{\theta\psi\gamma} &= \sin^2\theta \cos\theta \cos^2\phi \sin\gamma, \\
A_{\theta\omega\gamma} &= \sin^2\theta \cos\theta \sin^2\phi \sin\gamma, \\
A_{\phi\psi\theta} &= -\cos^3\theta \sin\phi \cos\phi \cos\gamma, \\
A_{\phi\psi\gamma} &= \cos^2\theta \sin\theta \sin\phi \cos\phi \cos\gamma, \\
A_{\phi\omega\theta} &= -\cos^3\theta \sin\phi \cos\phi \cos\gamma, \\
A_{\phi\omega\gamma} &= -\cos^2\theta \sin\theta \sin\phi \cos\phi \sin\gamma,
\end{aligned} \tag{3.22}$$

For the wave profile (H) we have

$$H = -\frac{\mu^2 q^{3/2}}{8u}\cos^2\theta \tag{3.23}$$

This $AdS_4 \times S^7$ gravitational wave solution, characterized by functions A_{ab}, A_{abc} and H in eqs (3.21), (3.22) and (3.23), is obtained from an $M2$ brane solution [12] in a pp-wave background as a near horizon geometry The $M2$ brane solution is given as

$$ds^2 = f^{-2/3}\left(2dx^+dx^- - H(dx^+)^2 + (dx)^2\right) + f^{1/3}\left(\sum_{i=1}^{8} dx_i^2\right),$$

$$F^{(4)} = dx^+ \wedge (\mu_1 dx_1 \wedge dx_2 + \mu_2 dx_3$$

$$\wedge dx_4 + \mu_3 dx_5 \wedge dx_6) \wedge dx_8,$$

$$f = \left(1 + \frac{q^3}{r^6}\right), \tag{3.24}$$

with H in equations (3.24) being

$$H = -\frac{\mu_1^2}{4}\left(x_1^2 + x_2^2\right) - \frac{\mu_2^2}{4}\left(x_3^2 + x_4^2\right) - \frac{\mu_3^2}{4}\left(x_5^2 + x_6^2\right) \tag{3.25}$$

4. Conclusion

In this article, we have reviewed the graviational wave solutions and some of their properties in pure gravity theory. We have also given similar solutions for string theory and M-theory The first class of such solutions are a simple embedding of the solutions in pure gravity, by simply turning on fluxes which compensate for the background curvature. This requirement follows from the fact that, unlike pure gravity, both these theories do not allow for the presence of a cosmological constant in the context that we are discussing Another class of solution, presented in Section 3, however, includes null fluxes with respect to p-form gauge field strengths These fluxes keep the background unchanged to $AdS_p \times S^q$ However, the wave-profiles are now modified, a fact which may possibly be exploited in any studies of primordial waves of this type In addition, it will be of interest to examine whether these null fluxes are associated with a wave propagation of p-form fields present in such theories In this context, to start with, it will be of interest to write down the analogs of the GDE for string and M-theory, for those objects which couple to the p-form fields, in addition to gravity The simplest possibility will then be to write down the GDE for the string propagation, coupling to antisymmetric tensor fields These will be then associated with new PP-signularities appearing due to the contraction of frame vectors with other field streetlights · $H^{(3)}_{\mu\nu\rho}$, $F^{(3)}_{\mu\nu\rho}$, etc

A holographic interpretation of the solutions is also possible In particular, restricting to five dimensional Kaigodorov solution, it is noticed that boundary does not change in the presence of gravitational waves This is because $H = u^4 \to 0$ on the boundary The holographic interpretation is then given in terms of a CFT with constant null momentum density . $P \equiv <T_{++}> \neq 0$ Symmetries of the CFT are also broken once the condition· $P = constant$ is imposed In this situation, an *AdS/CFT* interpretation is also given by realizing that Kaigodorov solution can be obtained from *AdS* by an infinite boost generating a constant null momentum density. In the new examples, given in this article, however, *CFT* interpretations are not obvious In fact, in some of our examples, H diverges at the *AdS* boundary, a property also shared by HPP-waves in flat space This prevents one from defining a boundary theory properly In certain other examples, with wave profiles of the type $H \sim \sum_i x^{i^2}$, though metric remains well defined at the *AdS* boundary, a proper study of the symmetries and classifications of states and operators are not fully understood, both in the bulk as well as boundary

To conclude, we have shown that it is possible to find generalized Siklos and Kaigodorov type solutions in string theory and M-theory, by turning on null fluxes We also argued that these new solutions are interesting to study from vaious angles

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